

Technical note

# A mathematical formula to calculate the theoretical range of motion for total hip replacement

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## Abstract

The reduced range of motion (ROM) resulting from total hip replacement (THR) leads to frequent prosthetic impingement, which may restrict activities of daily living and cause subluxation and dislocation. Therefore, to know the ROM of THR is very important in clinical situations and in the design of prostheses. THR involves a pure ball and socket joint. We created a mathematical formula to calculate the theoretical ROM of THR limited by the prosthetic impingement. The ROM of THR is governed by the following five factors, (1) The prosthetic ROM (oscillation angle: obtained from company data), (2) cup abduction (3) cup anterior opening, (4) the angle of the femoral neck component from the horizontal plane, and (5) the femoral neck anteversion. The last 4 factors are able to be obtained from anterior–posterior, axial X-rays and CT of the patient's THR. The objective was to create mathematical formulas that could accurately and quickly calculate the ROM of THR. By entering the five values into a computer programmed with the formulas, one could obtain the ROM for the THR. This reveals the effect on ROM of the oscillation angle and the interaction of ROM with cup abduction, anterior opening and neck anteversion. Furthermore this readily would enable a clinical evaluation of the possibility of postoperative dislocation and help in postoperative rehabilitation. The calculated numerical values of ROM by these mathematical formulas were successfully compared with the ROMs obtained from 3-dimensional computer graphics (3D-CG). © 2002 Elsevier Science Ltd. All rights reserved.

*Keywords:* Total hip replacement; Range of motion; Mathematical formula; Oscillation angle; Cup and neck orientation

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## 1. Introduction

Subjects with a normal hip joint have 120° flexion in common activities of daily living (ADL) (Johnson and Smidt, 1970). The reduced range of motion (ROM) resulting from total hip replacement (THR) leads to frequent prosthetic impingement, which may restrict ADL and cause subluxation and dislocation. It may also increase polyethylene debris and contribute to prosthetic loosening. Therefore, to know the ROM of THR is very important in both short- and long-term clinical situations and in the design of prostheses. Its ROM can be obtained by very expensive Computer Graphics: a 3D CAD model from each manufacturer of each of their total hip designs and sizes (3D-CG) or hip joint models, but it takes a lot of time and labor (Jaramaz et al., 1998; Kummer et al., 1999; Robinson et al., 1997; Scifert et al., 1998; Seki et al., 1998; D'Lima et al., 2000). Our objective is to create a mathematical formula that is able to calculate the ROM of THR in a very easy and accurate way, to reveal the effect on ROM of the oscillation angle and the interaction of ROM with cup abduction, anterior opening and neck anteversion.

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2. Materials and methods

The ROM of THR is governed by the following five factors. (1) Prosthetic ROM (oscillation angle),  $\theta$ . (2) Cup abduction,  $\alpha$ . (3) Cup anterior opening,  $\beta$ . (4) The angle of the neck position from the horizontal plane,  $a$ . (5) The anteversion of neck around the vertical axis (long body axis) from coronal plane,  $b$ .

The limit of the neck motion due to the impingement in the cup with a flat surface is described as a cone (the prosthetic ROM cone). The vertical angle of this cone is named as prosthetic ROM (oscillation angle,  $\theta$ ).

$$\theta = A - 2 \sin^{-1} \left( \frac{n/2}{r} \right) = A - 2 \sin^{-1} \left( \frac{1}{\text{head/neck ratio}} \right), \tag{1}$$

where  $A$  is the maximum angle of the radius movement in the cup,  $n$  is the neck width at the impingement level, and  $r$  is the radius of the head.

This angle depends upon the cup design and the head/neck ratio, so each prosthesis has its own oscillation angle. A larger head/neck ratio ( $2r/n$ ) creates a larger oscillation angle (Fig. 1a).

We defined the center of movement of the THR as the origin of coordinates, the body vertical axis (long body axis) as the  $Z$ -axis, the lateral-medial horizontal axis (transverse axis) as the  $Y$ -axis and the anterior–posterior horizontal axis (sagittal axis) as the  $X$ -axis (Figs. 1 and 2). The prosthetic ROM cone with a vertex at the origin facing downward is first rotated the amount of  $\alpha$  laterally about the  $X$ -axis, then it is rotated anteriorly the amount of  $\beta$  about the abducted axis. The abducted axis was defined as the cup diameter line through the origin on a coronal plane:  $Z = Y \tan \alpha$  (Fig. 1b). This cup anterior opening corresponds to the radiographic definition (Jaramaz et al., 1998). In this definition, the cup abduction angle  $\alpha$  does not change on the A–P view even after anterior opening  $\beta$  (Fig. 1b).

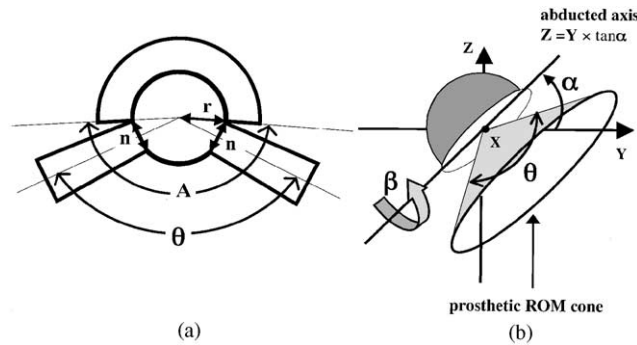


Fig. 1. The prosthetic range of motion cone: (a) the oscillation angle,  $\theta$ ; (b) left THR, cup anterior opening,  $\beta$  around the abducted axis after the cup initial abduction,  $\alpha$ .

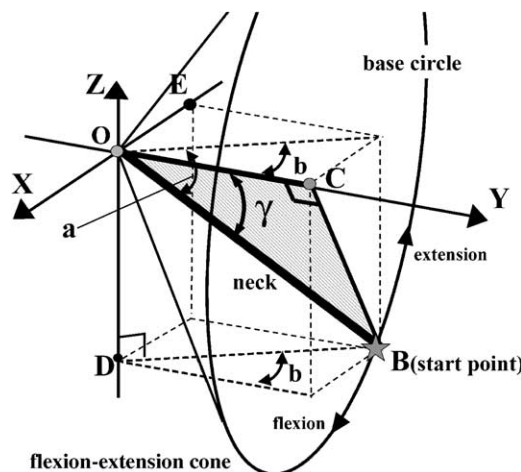


Fig. 2. Left THR, the flexion–extension cone of femoral neck component motion:  $C$  is the center of the base circle of the cone,  $B$  is the start position of the femoral neck component at the leg neutral position.  $a$  is true angle of the femoral neck component position from the horizontal  $X$ – $Y$  plane.  $b$  is anteversion of the femoral neck component around the vertical axis (long body axis) from coronal plane.  $\gamma$  is half of the vertical angle of the flexion–extension cone.  $\gamma = \cos^{-1}(\cos a \cos b)$ .

The neck of the femoral component during a flexion/extension leg movement follows the track of a cone with its vertex at the origin, vertical angle  $2 \times \gamma$  and the  $Y$ -axis as its axis (the flexion/extension cone,  $\gamma = \cos^{-1}(\cos a \cos b)$ ). The angle of the femoral neck component from the horizontal plane:  $a$  is the angle between the neck and  $X$ - $Y$  plane, and the anteversion of the femoral neck component:  $b$  is the angle between the neck and the  $Y$ - $Z$  plane (Fig. 2).

We consider both the prosthetic ROM cone and the flexion/extension cone as right circular cones with a unit length generatrix (slant height) and a base circle. Two intersection points between these two base circles are the impingement points of flexion and extension (L, M). We expressed these concepts mathematically by a vector analysis.

An equation of a unit sphere

$$x^2 + y^2 + z^2 = 1 \tag{2}$$

An equation:

$$x \times \sin \beta + y \times \sin \alpha \cos \beta - z \times \cos \alpha \cos \beta = \cos \frac{\theta}{2} \tag{3}$$

is created for the cone of the prosthetic ROM after abduction  $\alpha$  and anterior opening  $\beta$ .

An equation:

$$y = \cos a \cos b \tag{4}$$

is created for the flexion/extension cone.

Each equation (3) and (4) shows the base circle plane of each cone.

Combination of these three equations (2)–(4) forms the simultaneous quadratic equations with three unknowns ( $x, y, z$ ). The solutions of these equations are the coordinates of the two intersection points L and M. Then we devised the mathematical equations to calculate the range of flexion  $\angle LCB$  and extension  $\angle MCB$  until cup-neck impingement of the THR from the five factors ( $\theta, \alpha, \beta, a, b$ ), (Fig. 3). We can create the mathematical equations to calculate the ROM of the THR in any directions by the same method. We show the mathematical equations for the two extreme angles of 3 motion types (Figs. 3 and 4).

1. Flexion (FL) and extension (EXT) (Fig. 3)

$$E = \cos \frac{\theta}{2} - \sin \alpha \cos \beta \cos a \cos b, \quad D = (1 - \sin^2 \alpha \cos^2 \beta)(1 - \cos^2 a \cos^2 b) - E^2$$

Flexion:  $\angle LCB = FL$

$$FL = \cos^{-1} \left( \frac{(-\sin \beta \cos a \sin b + \cos \alpha \cos \beta \sin a)E - (\cos \alpha \cos \beta \cos a \sin b + \sin \beta \sin a)\sqrt{D}}{(1 - \sin^2 \alpha \cos^2 \beta)(1 - \cos^2 a \cos^2 b)} \right)$$

Extension:  $\angle MCB = EXT$

$$EXT = \cos^{-1} \left( \frac{(-\sin \beta \cos a \sin b + \cos \alpha \cos \beta \sin a)E + (\cos \alpha \cos \beta \cos a \sin b + \sin \beta \sin a)\sqrt{D}}{(1 - \sin^2 \alpha \cos^2 \beta)(1 - \cos^2 a \cos^2 b)} \right)$$

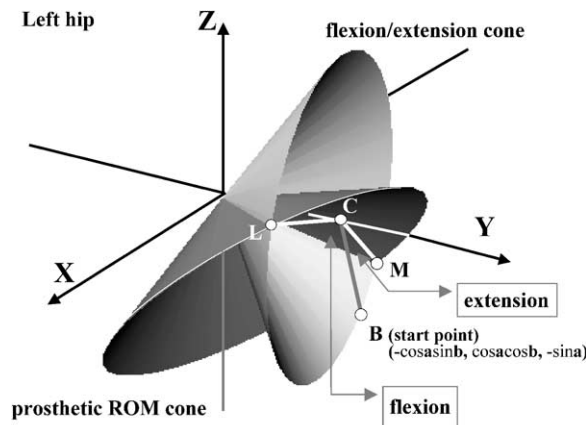


Fig. 3. Left THR, the prosthetic ROM cone and flexion/extension cone. L is flexion impingement point and M is extension impingement point.  $\angle LCB$  is flexion angle and  $\angle MCB$  is extension angle.

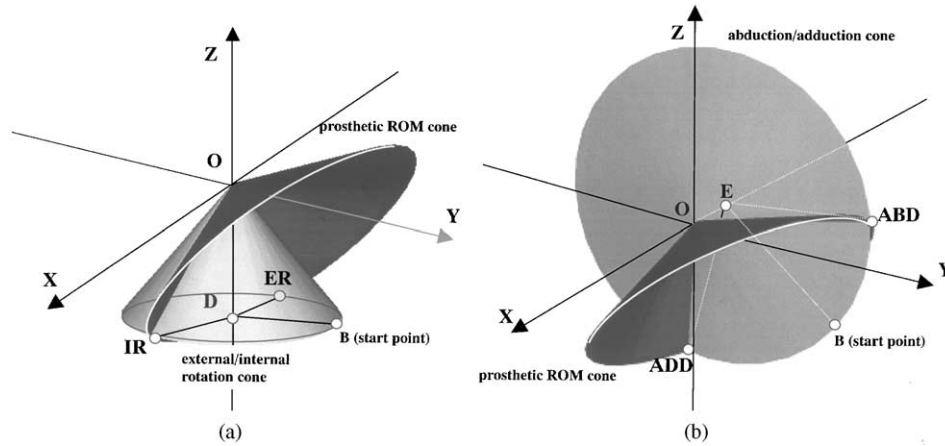


Fig. 4. Left THR (a) The prosthetic ROM cone and external/internal rotation cone. ER and IR are external rotation and internal rotation impingement points. (b) The prosthetic ROM cone and abduction/adduction cone. ABD and ADD are abduction and adduction impingement

## 2. External rotation (ER) and internal rotation (IR) (Fig. 4a)

$$E = \cos \frac{\theta}{2} - \cos \alpha \cos \beta \sin a, \quad D = (1 - \cos^2 \alpha \cos^2 \beta) \cos^2 a - E^2$$

External rotation: ER

$$ER = \cos^{-1} \left( \frac{(-\sin \beta \sin b + \sin \alpha \cos \beta \cos b)E + (\sin \alpha \cos \beta \sin b + \sin \beta \cos b)\sqrt{D}}{\cos a(1 - \cos^2 \alpha \cos^2 \beta)} \right)$$

Internal rotation: IR

$$IR = \cos^{-1} \left( \frac{(-\sin \beta \sin b + \sin \alpha \cos \beta \cos b)E - (\sin \alpha \cos \beta \sin b + \sin \beta \cos b)\sqrt{D}}{\cos a(1 - \cos^2 \alpha \cos^2 \beta)} \right)$$

## 3. Abduction (ABD) and adduction (ADD) (Fig. 4b)

$$E = \cos \frac{\theta}{2} + \sin \beta \cos a \sin b, \quad D = (1 - \cos^2 a \sin^2 b) \cos^2 \beta - E^2$$

Abduction: ABD

$$ABD = \cos^{-1} \left( \frac{(\sin \alpha \cos a \cos b + \cos \alpha \sin a)E + (\cos \alpha \cos a \cos b - \sin \alpha \sin a)\sqrt{D}}{\cos \beta(1 - \cos^2 a \sin^2 b)} \right)$$

Adduction: ADD

$$ADD = \cos^{-1} \left( \frac{(\sin \alpha \cos a \cos b + \cos \alpha \sin a)E - (\cos \alpha \cos a \cos b - \sin \alpha \sin a)\sqrt{D}}{\cos \beta(1 - \cos^2 a \sin^2 b)} \right)$$

How to make these equations and equations for extremes of additional motions are provided in Appendix A at Website of the Journal of Biomechanics: <http://www.elsevier.com/locate/jbiomech>.

The calculated numerical values of ROM for 16 different parameter sets of  $(\theta, \alpha, \beta, a, b)$  from our mathematical formula were compared to the ROM obtained from 3D-CG (Trilogy cup and Mutilock stem, Zimmer) made by Unigraphics Ver.4. The values obtained from both methods were identical within  $0.01^\circ$ . Thus, this simplified method of estimating the theoretical ROM appears to be as accurate as the complex method of 3D-CG.

## 3. Results

1. A  $\theta$  greater than  $120^\circ$  seems to be necessary to fulfill an acceptable ROM. When  $a$  was fixed to  $52^\circ$  and  $b$  was fixed to  $15^\circ$ , FL and ER were calculated for different oscillation angles ( $\theta = 100^\circ$ – $135^\circ$ ) in three cup positions ( $\alpha = 35^\circ, \beta = 10^\circ$ ), ( $\alpha = 45^\circ, \beta = 20^\circ$ ), ( $\alpha = 55^\circ, \beta = 30^\circ$ ). A  $120^\circ$  oscillation angle gives only  $99^\circ$  FL and  $80^\circ$  ER in ( $\alpha = 35^\circ, \beta = 10^\circ$ ),  $139^\circ$  FL and only  $28^\circ$  ER in ( $\alpha = 55^\circ, \beta = 30^\circ$ ), and  $119^\circ$  FL and  $51^\circ$  ER even in ( $\alpha = 45^\circ, \beta = 20^\circ$ ).

2. We created the 3D graphs showing the interaction of  $\alpha$  and  $\beta$  to FL, ER and EXT when  $\theta = 120^\circ$ ,  $a = 52^\circ$ ,  $b = 20^\circ$ , and the interaction of  $\beta$  and  $b$  to FL, ER and EXT when  $\theta = 120^\circ$ ,  $\alpha = 45^\circ$ ,  $a = 52^\circ$  (See Appendix B at Website). When each value of  $\alpha$ ,  $\beta$  and  $b$  increases, FL and ER increase. When each value of  $\beta$  and  $b$  increases, EXT increases.  $\beta$  is the most influential factor in FL, ER and EXT.

#### 4. Discussion

These formulas are a very accurate, fast and easy way to get the ROM for a THR with a flat surface cup and a symmetric neck. The oscillation angle is taken from the company data or measured from a blueprint of the cup, head and neck shape. The values of  $\alpha$  and  $\beta$  of the cup position and  $a$  and  $b$  of the neck position can be measured or calculated from an accurate anterior–posterior X-ray view, and an axial CT view of the THR (see Appendix C at Website).

Lewinnek proposed a cup position with abduction  $30\text{--}50^\circ$  and anterior opening  $5\text{--}25^\circ$  as a cup safe area (Lewinnek et al., 1978), however this has no theoretical support. Even  $120^\circ$   $\theta$  gives only  $99^\circ$  FL in cup position ( $\alpha = 35^\circ$ ,  $\beta = 10^\circ$ ), which is inside the safe area. ROM in any direction is directly related not to the head/neck ratio but to  $\theta$ , and ROM almost linearly increases by  $\theta$ . D’Lima showed that an increase in head size of 4 mm, from 22 to 26 mm, resulted in a larger improvement in ROM than did a increase from 28 to 32 mm using 3D-CG model, however he does not mention the reason (D’Lima et al., 2000). The head size of 22, 26, 28, and 32 mm has the head/neck ratio of 1.8, 2.1, 2.3, and 2.6, and  $\theta$  of  $112^\circ$ ,  $124^\circ$ ,  $128^\circ$ , and  $135^\circ$ , respectively (Eq. (1)). The differences of  $\theta$  are  $12^\circ$  for 22–26 mm heads and  $7^\circ$  for 28–32 mm heads. This is the reason for his results. Seki et al. made a 3D ROM graph showing the interaction of  $\alpha$  and  $\beta$  with FL, and EXT by a 3D-CG model. They neither show ER, nor the interaction of  $\beta$  and  $b$ . It would take a lot of time and labor to analyze ROM by 3D-CG model, however we can predict the ROM of THR almost instantly by feeding the data of the 5 factors into a computer programmed with the formulas (within a second by Excel 97 of Windows 98 of an ordinary computer with CPU Pentium 233 MHz).

Using these formulas, we can analyze the characteristics of THR, (the complicated interaction of the 5 factors ( $\theta$ ,  $\alpha$ ,  $\beta$ ,  $a$ ,  $b$ ) to ROM, a meaningful cup safe area based on ROM, the oscillation angle needed for no impingement, the impingement position on the cup edge, a behavior of the oblique cup etc.). The ROM calculated by these formulas for each patient would be very helpful for preventing postoperative dislocation as well as its clinical evaluation.

When the ROM before cup neck impingement is small, the possibility of subluxation or dislocation is large. Although this study takes no account of bony or soft tissue impingement, in the case of bony or soft tissue impingement, the ROM decreases much more because it occurs before cup neck impingement, so that the possibility of subluxation or dislocation becomes much greater. Therefore, it is at least necessary that the ROM before cup neck impingement be large enough.

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